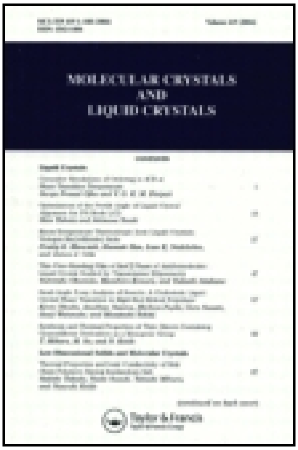

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Calculations of Electric Field Dependence of Effective Refractive Index in Nematic Liquid Crystal Panel

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Novel approach to calculation of changes of extraordinary effective index of refraction upon action of an external electric field for a case of a homogeneous nematic is presented. The solution of the static sine-Gordon equation, even for the simplest case (symmetrical hard boundary conditions, one elastic constant approximation within Ericksen–Lesile theory) requires an use of Jacobian elliptic functions. The numerical calculations based on the presented solutions agree well with the experimental data.

Keywords: Nematic liquid crystal; effective index of refraction; electric Fréedericksz splay effect
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1. INTRODUCTION

Processing of optical images or matrix data as 2-D slices of information can be vastly more efficient than the time-sequential (bit-by-bit) processing used in conventional computers. At present, liquid crystal spatial light modulators are frequently used for that purpose. In electrically (active matrix displays) or optically addressed nematic liquid crystal spatial light modulators [1–4] the precise control of an effective index of refraction is crucial for operation of the device. This control is achieved by proper application of an electric voltage to the liquid crystal cell. This voltage change the effective index of refraction, which in turn has its impact on a

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light diffraction efficiency or a phase retardation in liquid crystal devices. However, the effective index of refraction, seen by light traversing the liquid crystal panel, is in fact a complicated function of many physical parameters of liquid crystal itself, strength and distribution of external electric or magnetic fields, as well as geometry, the strength of surface anchoring forces and finally a light polarization and its propagation directions. All these problems were separately addressed in many papers and were presented in several textbooks on liquid crystals [5, 6].

In this paper we will present a novel, up to our knowledge, solution of the equation for effective index of refraction for a particular case of planar nematic liquid crystal cell (panel) under action of a static, external field **E** (*i.e.*, the well known electric Fréedericksz effect or electrically controlled birefringence [5, 6]).

2. STATEMENT OF THE PROBLEM

Let us consider a nematic liquid crystal panel consisting of two parallel glass windows separated by a small distance L (typically 15 μm). A nematic liquid crystal is filling the cell. In the absence of either electric **E** or magnetic **H** fields, the nematic director $\hat{n}$ orientation is determined by specific molecular interactions and molecular interactions with the rubbed surfaces, which impose ordering. We will examine the case of planar (homogeneous) liquid crystal panel with assumption of the hard boundary conditions. The latter are equivalent to an assumption that the molecules are strongly anchored to the surfaces with their long axes parallel to the rubbing direction *i.e.*, along the x-axis at positions $z=0$ and $z=L$, respectively.

Schematic diagram of the liquid crystal cell and adopted coordinate axes system is depicted in Figure 1. For the sake of simplicity we will consider only the static electric field along the z-axis, $\mathbf{E}=E(0\ ,0\ ,1)$. Such a field causes an in-plane (x, z) reorientation of the long molecular axes. Angular distribution function $\varphi(z)$ describes the director $\hat{n}$ dependence on the distance along the LC layer thickness:

$$\hat{n} = (\cos\varphi(z),\ 0,\ \sin\varphi(z)), \tag{1}$$

where $\varphi(z)$ is the angle $\hat{n}$ makes with the x-axis. The refractive index seen by the extraordinary polarized light beam incident normally to the surface of the planar nematic liquid crystal panel with uniform distribution of director

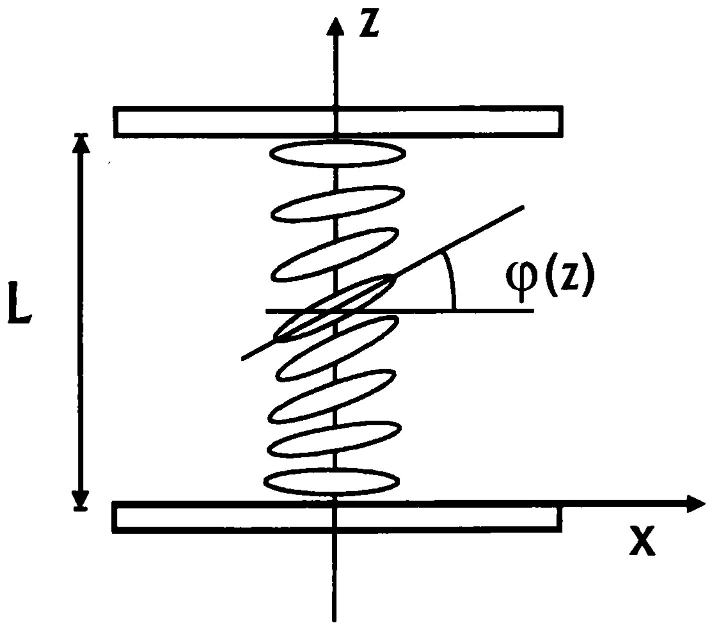

FIGURE 1 Schematic view of homogenous (planar) nematic liquid crystal cell with the coordinate system. Director orientation due to voltage applied to electrodes is described by the angle $\varphi(z)$.

orientation angles $\hat{n}(\varphi)$ is given by the formula [6]:

$$n_e^{\mathrm{eff}} = \frac{n_e n_o}{\sqrt{n_e^2 \sin^2 \varphi + n_o^2 \cos^2 \varphi}}, \tag{2}$$

where n_e and n_o are extraordinary and ordinary refractive indices, respectively.

In the case of nematic liquid crystal interacting with an external electric field applied along the z-axis (a classical electric Fréedericksz splay effect [5]), the molecules turn in the xz-plane according to the torque which is the result of a balance between the electric and elastic forces. A local value of director angle φ is a function of position z with respect to the sample thickness L. This causes that for any boundary conditions there is an index profile across the sample thickness. Extraordinary polarized light beam propagating across the sample thickness "sees" an average effective index of refraction $\langle n_e^{\mathrm{eff}} \rangle$, defined by the formula [5]:

$$\langle n_e^{\mathrm{eff}} \rangle = \frac{1}{L}\int_0^L \frac{n_e n_o dz}{\sqrt{n_e^2 \sin^2 \varphi(z) + n_o^2 \cos^2 \varphi(z)}} = \frac{n_e}{L}\int_0^L \frac{dz}{\sqrt{1 + a \sin^2 \varphi(z)}}, \tag{3}$$

where $a = (n_e^2 - n_o^2)/n_o^2$. Below we will present the solution for an effective extraordinary index of refraction (3) using the Ericksen – Leslie theory of nematics [7]. Within this theory the form of $\varphi(z)$ function can be found.

3. CALCULATIONS OF ANGULAR DISTRIBUTION OF DIRECTOR $\varphi(z)$

According to Ericksen – Lesile theory [7] the equation describing direction orientation for the static field case can be written in the form:

$$0 = K_F \frac{d^2\varphi(z)}{dz^2} + \frac{1}{2}\varepsilon_0 \Delta\varepsilon \sin(2\varphi(z))E^2 \tag{4}$$

where $\Delta\varepsilon = \varepsilon_{\parallel} - \varepsilon_{\perp} > 0$ is the dielectric anisotropy, ε_0 is the permittivity of free space and K_F is Frank elastic constant. We restricted ourselves to the case of single elastic constant approximation *i.e.*, when $K_{11} = K_{22} = K_{33} = K_F$. Equation (4) describes the director orientation profile above a so called Fréedericksz threshold field E_{th}:

$$E_{th} = \sqrt{\frac{K_F}{\varepsilon_0 \Delta\varepsilon}} \frac{\pi}{L} \tag{5}$$

Equation (4) is a version of the static sine-Gordon equation which can be solved under specific boundary conditions. Assumption of the hard boundary conditions $\varphi(0) = \varphi(L) \equiv 0$ forces the center of the director re-orientation to occur at $z = L/2$ with $(d\varphi/dz)|_{z=L/2} = 0$. Now the Eq. (4) can be integrated and rewritten as:

$$\left(\frac{d\varphi(z)}{dz}\right)^2 = -\frac{\varepsilon_0 \Delta\varepsilon}{K_F} \sin^2 \varphi(z) E^2 + \left(\left.\frac{d\varphi(z)}{dz}\right|_{z=0}\right)^2 \tag{6}$$

Integration of the Eq. (6) leads to the integral of Jacobian elliptic function:

$$\int_0^{\varphi(z)} \frac{d\tau}{\sqrt{1 - m \sin^2 \tau}} = z \left.\frac{d\varphi(z)}{dz}\right|_{z=0}, \tag{7}$$

where $\varphi(z)$ is called the amplitude and the parameter m is given by the expression:

$$m = \frac{\varepsilon_0 \Delta\varepsilon E^2}{K_F \left(\left.\frac{d\varphi(z)}{dz}\right|_{z=0}\right)^2}. \tag{8}$$

Then, the solution of the Eq. (5) can be obtained in terms of Jacobian elliptic functions [8]. Rewriting Eq. (7) in the form:

$$u = \int_0^{\varphi(z)} \frac{d\tau}{\sqrt{1 - m\sin^2\tau}} \tag{9}$$

and making use of the Jacobian function sn(u/m):

$$\mathrm{sn}(u|m) = \sin\varphi(z) \tag{10}$$

we can find the solution for $\varphi(z)$ assuming that the parameter m fulfills $0 \le m \le 1$. However, in our case $m = (1/\sin^2\varphi(L/2)) \ge 1$, which follows from Eq. (6) with $z = L/2$. Then, it is advised to introduce a new parameter $\mu = m^{-1} < 1$ with appropriate Jacobi's real transformation:

$$\mathrm{sn}(u|m) = \mu^{(1/2)}\mathrm{sn}(v|\mu) = \sin\varphi(z) \tag{11}$$

where $v = um^{1/2}$. Now, expanding sn($v|\mu$) with respect to the nome q and the argument s [8] we arrive at the expression:

$$\mathrm{sn}(v/\mu) = \frac{2\pi}{\mu^{1/2}K(\mu)} \sum_{n=0}^{\infty} \left(\frac{q^{n+1/2}}{1 - q^{2n+1}} \right) \sin[(2n+1)s] \tag{12}$$

where

$$q = \exp\left(-\frac{\pi K'(\mu)}{K(\mu)} \right),$$

$$K(\mu) = \int_0^{\pi/2} \frac{d\tau}{\sqrt{1 - \mu\sin^2\tau}},$$

$$K'(\mu) = \int_0^{\pi/2} \frac{d\tau}{\sqrt{1 - (1-\mu)\sin^2\tau}}$$

and

$$s = \frac{\pi v}{2K(\mu)}.$$

One can easily show (Eqs. (8) and (9)) that the argument s of this expansion is expressed by:

$$s = \frac{\pi v}{2K(\mu)} = \frac{\pi z E}{2K(\mu)} \sqrt{\frac{\varepsilon_0 \Delta\varepsilon}{K_F}}. \tag{13}$$

Substituting Eq. (12) into (11) one obtains the functional dependence of profile of reorientations angles along the sample thickness $\varphi(z)$ in the form:

$$\sin\varphi(z) = \frac{2\pi}{K(\mu)}\sum_{n=0}^{\infty}\left(\frac{q^{n+1/2}}{1-q^{2n+1}}\right)\sin\left[(2n+1)\sqrt{\frac{\varepsilon_0\Delta\varepsilon}{K_F}}\frac{\pi zE}{2K(\mu)}\right], \tag{14}$$

where n is an integer number. Numerical calculation of function $\varphi(z)$ can be performed but it requires a very precise determination of q.

For the purpose of this paper we use the Eq. (14) for determination of $K(\mu)$. Using the condition $(d\varphi/dz)|_{z=L/2}=0$, which is fulfilled from symmetry reasons and applying it to derivative of Eq. (14), one obtains:

$$\begin{aligned}\left.\frac{d\sin\varphi(z)}{dz}\right|_{z=(L/2)} &= \left.\frac{d\sin\varphi(z)}{d\varphi(z)}\right|_{z=(L/2)}\left.\frac{d\varphi(z)}{dz}\right|_{z=(L/2)} = \\ &= -\frac{2\pi}{K(\mu)}\sum_{n=0}^{\infty}\left(\frac{q^{n+1/2}}{1-q^{2n+1}}\right)\left((2n+1)\sqrt{\frac{\varepsilon_0\Delta\varepsilon}{K_F}}\frac{\pi E}{2K(\mu)}\right)\times \\ &\quad\times\cos\left[(2n+1)\sqrt{\frac{\varepsilon_0\Delta\varepsilon}{K_F}}\frac{\pi LE}{4K(\mu)}\right]=0\end{aligned} \tag{15}$$

The Eq. (15) is satisfied when an argument for cosine function is equal to an odd number of $\pi/2$. Therefore one can arrive at condition:

$$\sqrt{\frac{\varepsilon_0\Delta\varepsilon}{K_F}}\frac{\pi LE}{4K(\mu)}=\frac{\pi}{2}.$$

In this way we have obtained the simple relation for $K(\mu)$:

$$K(\mu) = K\left(\frac{1}{m}\right) = \sqrt{\frac{\varepsilon_0\Delta\varepsilon}{K}}\frac{LE}{2}, \tag{16}$$

which will be used further in this work. It is worthwhile to mention that Eq. (16) allows one to easily calculate the threshold Fréedericksz field E_{th}. Noting that $K(\mu)$ (*cf.* Eq. (12)) for positive values of $0 \le \mu < 1$ can be expressed as:

$$K(\mu) = \int_0^{\pi/2}\frac{d\tau}{\sqrt{1-\mu\sin^2\tau}} \ge \frac{\pi}{2} \tag{17}$$

and using the relation (16) one can obtain the condition:

$$\sqrt{\frac{\varepsilon_0 \Delta\varepsilon}{K}} \frac{LE}{2} \geq \frac{\pi}{2}.$$

With a value of electric field E equal to the threshold Fréedericksz field E_{th} the above expression is equal to $\pi/2$. In this way we demonstrate the validity of the well known relation (5). Moreover, Eq. (17) allows one to calculate the quarter-periods $K(\mu)$ and $K'(\mu)$ for any homogeneously aligned nematic liquid crystal characterized by a Frank elastic constant K_F and a dielectric anisotropy $\Delta\varepsilon$.

4. DERIVATION OF EXTRAORDINARY REFRACTIVE INDEX DEPENDENCE ON ELECTRIC FIELD STRENGTH

In order to calculate the average extraordinary index of refraction $\langle n_e^{\text{eff}} \rangle$ (without introducing the whole expression (14) for $\varphi(z)$) one can apply the proposed by us formalism. According to this formalism the Eq. (3) can be rewritten using the relation (10) as:

$$\langle n_e^{\text{eff}} \rangle = \frac{n_e}{L} \int_0^L \frac{dz}{\sqrt{1 + a\, \text{sn}^2(u(z)|m)}}. \tag{19}$$

The above integral can be further transformed with use of Jacobian elliptic functions and their properties [8]. Taking the following relations quoted in Ref. [8]:

$$\frac{d\,\text{sn}(u(z)|m)}{d\,u(z)} = \text{cn}(u(z)|m)\ \text{dn}(u(z)|m) \tag{20}$$

where $\text{cn}(u(z)|m) = \cos\varphi(z)$ and $\text{dn}(u(z)|m) = \sqrt{1 - m\ \sin^2 \varphi(z)}$ with the relation (8) the following form of the integral (19) can be derived:

$$\langle n_e^{\text{eff}} \rangle = 2\frac{n_e}{LE}\sqrt{\frac{K_F}{\varepsilon_0 \Delta\varepsilon}} \int_0^{\pi/2} \frac{d\tau}{\sqrt{(1 + a\mu \sin \tau^2)(1 - \mu \sin^2 \tau)}} \tag{21}$$

Making use of the relation (16) for $K(\mu)$ and the definition of the quarterperiod itself (*cf.* Eq. (12)) one finally obtains the relation for effective index of refraction:

$$\langle n_e^{\text{eff}}\rangle = n_e \frac{\int_0^{\pi/2} \frac{d\tau}{\sqrt{(1+a\mu\sin\tau^2(1-\mu\sin^2\tau)}}}{\int_0^{\pi/2} \frac{d\tau}{\sqrt{1-\mu\sin^2\tau}}} = \tag{22}$$

$$= n_e \frac{\int_0^{\pi/2} \frac{d\tau}{\sqrt{(1+a\mu\sin\tau^2)(1-\mu\sin^2\tau)}}}{K(\mu)} = n_e F(K(\mu)).$$

Defined by this expression function $F(K(\mu))$ depends on material and cell parameters through parameters: a and μ. Relation (22) is particularly convenient for calculations due to a linear dependence between $K(\mu)$ function and electric field strength E (*cf.* Eq. (16)). This allows for an easy calculation of an effective extraordinary index of refraction providing the necessary parameters L, K, $\Delta\varepsilon$, n_e, n_o and E are given. This dramatically simplifies the calculations of average $\langle n_e^{\text{eff}}\rangle$ as given by formula (3) because avoids problem of exact knowledge of angular director distribution across the cell thickness. Notice that the Eq. (22) allows one to describe all liquid crystal panels, characterized by different parameters L, K_F and $\Delta\varepsilon$ but having the same value of parameter a (*cf.* Eq. (3)), with a single functional dependence $F(K(\mu))$. In other words all the curves $\langle n_e^{\text{eff}}\rangle$ (E) with a common value of parameter a can be superimposed on each other by simple rescalling at least within the Ericksen – Lesile theory and under the above mentioned assumptions. The abscissa should be rescaled by a factor $1/n_e$ and ordinate axis by a factor

$$\sqrt{\frac{\varepsilon_0 \Delta\varepsilon}{K_F}}\frac{L}{2} = \frac{\pi}{2E_{\text{th}}}.$$

In Figure 2 we present the function $F(K(\mu))$ *versus* $K(\mu)$ for three different values of a parameter $a = 0.28$, 0.71 and 1.13. The presented curves can be directly compared with the experimentally determined electric field dependencies of extraordinary index of refraction for a planar nematic cell [9]. $F(K(\mu))$ function is proportional to the value of index of refraction while its argument $K(\mu)$ is proportional to the value of electric field strength acting on nematic and is always higher than $\pi/2$ ($\pi/2$ corresponds to E_{th}). As an example let us evaluate the value of extraordinary index of refraction for a planar nematic cell having the Fréedericksz voltage threshold $U_{\text{th}} =$

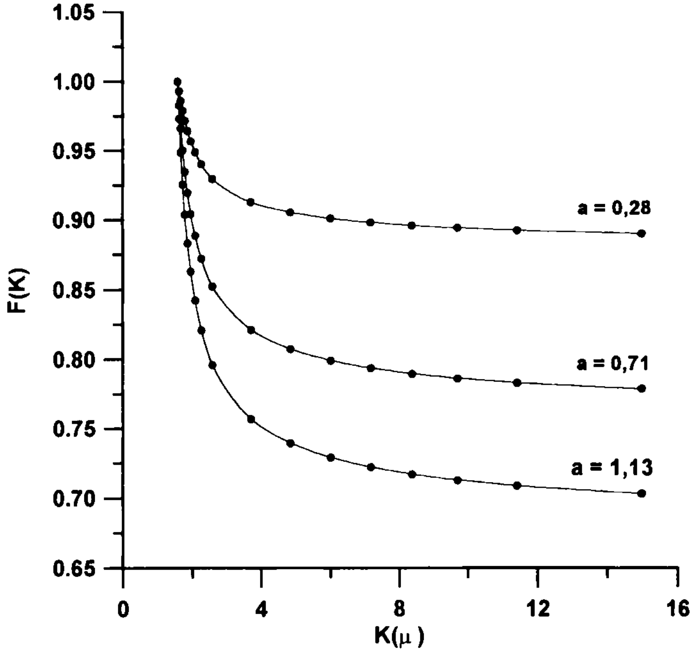

FIGURE 2 Plot of the function $F(K(\mu))$ *versus* $K(\mu)$ for three different values of a parameter $a=0.28$, 0.71 and 1.13.

$E_{\text{th}}L = 1.5\text{V}$ and $n_e = 1.72$, $n_o = 1.52$ then $a = 0.28$. Using the presented by us curve, calculated for this parameter $a = 0.28$, one can estimate the value of $\langle n_e^{\text{eff}} \rangle$ for a voltage equal $U = 2U_{\text{th}} = 3\text{V}$. In that case the value of a quarterperiod $K(\mu) = \pi$ and calculated value of $F(K(\mu)) = 0.92$. This allows us to obtain the value of $\langle n_e^{\text{eff}} \rangle = n_e \times F(K(\mu)) = 1.72 \times 0.92 = 1.582$. This result is very close to that reported for a homogeneous nematic liquid crystal panel reported by Klaus *et al.* [9] at the same voltage and equal to 1.58. Comparison of the theoretically obtained by us value with a real experimentally measured value is satisfactory.

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